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The stability of the equilibrium positions of non-linear non-autonomous mechanical systems☆

A.Yu. Aleksandrov

St. Petersburg, Russia

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Abstract

The problem of the stability of the equilibrium positions for a certain class of non-linear mechanical systems under the action of time-dependent quasipotential and dissipative-accelerating forces is considered. A method is proposed for constructing Lyapunov functions for these systems. Sufficient conditions for the stability of an equilibrium position both with respect to all of the variables as well as with respect to some of the variables are determined using the direct Lyapunov method.

1. Formulation of the problem

Suppose the motion of a mechanical system is described by the equations

$$\ddot{\mathbf{x}} + \mathbf{A}(t)\dot{\mathbf{x}} + \mathbf{B}(t)G_{\mathbf{x}} = \mathbf{0} \tag{1.1}$$

Here $\mathbf{x}$ is an n-dimensional vector, the matrix $\mathbf{A}(t)$ is definite and continuous when $t \geq 0$, $\mathbf{B}(t)$ is a symmetric matrix which is continuously differentiable when $t \geq 0$, the scalar function $G(\mathbf{x})$ is specified and is doubly differentiable when $||\mathbf{x}|| < H$ (H is a positive constant), $||\cdot||$ is the Euclidean norm of the vector and $G_{\mathbf{x}} = \partial G/\partial \mathbf{x}$. We shall assume that the matrix $\mathbf{B}(t)$ is positive-definite, that is, a number $b_0 > 0$ exists such that the following inequality holds for all $t \geq 0$ and $\mathbf{x} \in \mathbf{E}^n$

$$\mathbf{x}^T\mathbf{B}(t)\mathbf{x} \geq b_0\|\mathbf{x}\|^2$$

We will also assume that $G(\mathbf{x})$ is a positive-definite function and $G_{\mathbf{x}} \neq 0$ when $\mathbf{x} \neq 0$. System (1.1) then has an isolated equilibrium position

$$\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0} \tag{1.2}$$

We will investigate the stability of this equilibrium position. The problem of the stability of the equilibrium positions of non-autonomous mechanical systems has been studied in many papers (Refs. 1–9, for example) The basic method for solving this problem is the method of Lyapunov functions and its various modifications.

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E-mail address: alex@vrm.apmath.spbu.ru.

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Sufficient conditions for the stability of the equilibrium position (1.2) of system (1.1) both with respect to all of the variables as well as with respect to some of the variables are subsequently determined. Here, the approaches used in Refs. 1,3,6 are used and are further developed.

2. Construction of Lyapunov functions with a sign-definite derivative

The stability of the equilibrium positions of mechanical systems under the action of time-dependent forces was investigated in Refs. 6,10–12 using the method of limiting functions and limiting equations. The results obtained enable us to formulate the sufficient conditions for the asymptotic stability of the equilibrium position (1.2) of system (1.1). However, it should be pointed out that the approach which was used in Refs. 6,10–12 is based on the construction of Lyapunov functions for the systems being investigated which have sign-constant derivatives. A serious drawback of this approach is the fact that it does not enable one to determine the conditions for the preservation of asymptotic stability when the parameters of the systems being considered are known with a certain error or when external perturbing forces are acting on the system. It is well known (see Ref. 13, p.97) that, in order to obtain the above-mentioned conditions, it is only possible to apply Lyapunov theorems in which a sign-definite derivative is allowed.

The main purpose of this section of the paper is to show that the sufficient conditions for the asymptotic stability of an equilibrium position, found in Refs. 6,10–12 can be obtained using Lyapunov functions, the derivative of which, by virtue of system (1.1), is negative-definite.

The conditions for the asymptotic stability of the linear oscillator

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0 \tag{2.1}$$

were investigated in Ref. 3, where $x = x(t)$ is an unknown scalar function and the function $a(t)$ is continuous and bounded when $t \geq 0$. A method was proposed for constructing a Lyapunov function for system (2.1) which satisfies the requirements of Lyapunov's theorem on asymptotic stability (Ref. 2, pp. 30, 31). We shall consider the extension of this method to a system of the form of (1.1).

Suppose the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are bounded in the interval $[0, +\infty)$. In accordance with the approach proposed earlier in Ref. 6, the Lyapunov function for system (1.1) is chosen in the form

$$V = G(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}} \tag{2.2}$$

We assume that the matrix

$$\mathbf{C}(t) = \dot{\mathbf{B}}(t) + \mathbf{A}(t)\mathbf{B}(t) + \mathbf{B}(t)\mathbf{A}^T(t) \tag{2.3}$$

is positive-definite. It is well known[6] that the equilibrium position (1.2) of system (1.1) is then asymptotically stable. In this case, $dV/dt|_{(1.1)} \leq 0$.

We next consider the function

$$V_1 = G(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}} + G(\mathbf{x})\dot{\mathbf{x}}^T G_{\mathbf{x}} \tag{2.4}$$

We now differentiate this function by virtue of system (1.1). We have

$$\left.\frac{dV_1}{dt}\right|_{(1.1)} = -\frac{1}{2}\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\mathbf{C}(t)\mathbf{B}^{-1}(t)\dot{\mathbf{x}} - G(\mathbf{x})G_{\mathbf{x}}^T\mathbf{B}(t)G_{\mathbf{x}} + \\ + \dot{\mathbf{x}}^T(G(\mathbf{x})G_{\mathbf{x}})_{\mathbf{x}}\dot{\mathbf{x}} - G(\mathbf{x})\dot{\mathbf{x}}^T\mathbf{A}^T(t)G_{\mathbf{x}}$$

Taking account of the boundedness of the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ in the interval $[0, +\infty)$ and the positive definiteness of the matrices $\mathbf{B}(t)$ and $\mathbf{C}(t)$, we obtain that the limits

$$\begin{gathered} \alpha_1(G(\mathbf{x}) + \|\dot{\mathbf{x}}\|^2) \leq V_1 \leq \alpha_2(G(\mathbf{x}) + \|\dot{\mathbf{x}}\|^2) \\ \left.\frac{dV_1}{dt}\right|_{(1.1)} \leq -\alpha_3(G(\mathbf{x})\|G_{\mathbf{x}}\|^2 + \|\dot{\mathbf{x}}\|^2) \end{gathered} \tag{2.5}$$

hold for all $t \geq 0$, $\mathbf{x} \in \mathbf{E}^n$ and sufficiently small values of $||\mathbf{x}||$.

Here $\alpha_1, \alpha_2, \alpha_3$ are positive constants. The function (2.4) therefore satisfies the requirements of Lyapunov's theorem on asymptotic stability.

Remarks.

1°. The introduction of the additional term $G(\mathbf{x})\dot{\mathbf{x}}^T\ G_\mathbf{x}$ into the Lyapunov function ensures the negative definiteness of its derivative by virtue of the system being studied. Similar methods for converting Lyapunov functions with sign-negative derivatives into functions with negative definite derivatives[14–18] have been used when investigating the stability of autonomous mechanical systems.

2°. It will be shown later that, when solving certain problems, it is necessary to correct the form of the additional term in the Lyapunov function with the aim of obtaining the most effective results. However, in this case, additional assumptions are required regarding the functions occurring in the equations being investigated.

3°. The conditions of asymptotic stability obtained using Lyapunov function (2.4) are identical to the conditions established earlier.[6,10–12] However, the method, proposed in this paper, for constructing a Lyapunov function which satisfies the requirements of Lyapunov's theorem on asymptotic stability enables us not only to prove the asymptotic stability of an equilibrium position but also to obtain estimates of the decay time of transients and determine the conditions under which, when they are satisfied, asymptotic stability is also preserved in the case of perturbed systems.

4°. On the other hand, Lyapunov function (2.2), proposed earlier in Ref. 6, possesses a number of advantages. For example, more accurate estimates of the domain of asymptotic stability of the equilibrium position are obtained using it and, in the case when $G(\mathbf{x}) \to +\infty$ when $||\mathbf{x}|| \to \infty$, it can be proved that an equilibrium position is stable on the whole. The Lyapunov function constructed using formula (2.4) does not allow one to do this since the estimates (2.5), which are established for a given function and its derivative have a local character (they are satisfied for fairly small values of $||\mathbf{x}||$). Hence, to solve some problems, it turns out to be more effective to use the Lyapunov function proposed earlier in Ref. 6 and, for other problems, it is better to use the Lyapunov function with a negative-definite derivative which has been constructed above.

We shall next assume that the parameters of the system being investigated are known with a certain error. Together with Eq. (1.1), we will now consider the perturbed equations

$$\ddot{\mathbf{x}} + (\mathbf{A}(t) + \tilde{\mathbf{A}}(t))\dot{\mathbf{x}} + (\mathbf{B}(t) + \tilde{\mathbf{B}}(t))G_\mathbf{x} = \mathbf{0} \tag{2.6}$$

The matrices $\tilde{\mathbf{A}}(t)$ and $\tilde{\mathbf{B}}(t)$ are continuous for all $t \geq 0$ and satisfy the inequalities

$$\|\tilde{\mathbf{A}}(t)\| \leq \Delta_1, \quad \|\tilde{\mathbf{B}}(t)\| \leq \Delta_2$$

where Δ_1 and Δ_2 are positive constants.

As before, we shall assume that the matrix $\mathbf{C}(t)$ is positive-definite and that the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are bounded when $t \geq 0$. With these assumptions, positive constants a_0, a_1, b_1, b_2 exist such that the estimates

$$\mathbf{x}^T\mathbf{C}(t)\mathbf{x} \geq a_0\|\mathbf{x}\|^2, \quad \|\mathbf{A}(t)\| \leq a_1$$
$$\mathbf{x}^T\mathbf{B}^{-1}(t)\mathbf{x} \geq b_1\|\mathbf{x}\|^2, \quad \|\mathbf{B}^{-1}(t)\| \leq b_2$$

hold for all $t \geq 0$ and $\mathbf{x} \in \mathbf{E}^n$.

Moreover, we shall assume that the function $G(\mathbf{x})$ satisfies the conditions

$$\partial^2 G(\mathbf{0})/\partial\mathbf{x}^2 = \mathbf{0}, \quad \|G_\mathbf{x}\|^2/G(\mathbf{x}) \to 0 \text{ when } \|\mathbf{x}\| \to 0$$

We will now show that the proposed method for constructing Lyapunov functions enables one to obtain estimates of the permissible variations in the parameters of a system for which the asymptotic stability of the equilibrium position is preserved.

Theorem 1. *When the inequality*

$$2b_0b_2\Delta_1 + (2a_1b_2 + a_0b_1^2)\Delta_2 < a_0b_0b_1^2 \tag{2.7}$$

is satisfied, the equilibrium position (1.2) of system (2.6) is asymptotically stable.

Proof. We choose the Lyapunov function in the form

$$V_2 = G(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}} + \theta\dot{\mathbf{x}}^T G_{\mathbf{x}} \tag{2.8}$$

where θ is a positive constant.

For any $\theta > 0$, a neighbourhood of the point $\mathbf{x} = \mathbf{0}$ exists such that, when $t \geq 0$, the estimate

$$V_2 \geq \frac{1}{2}G(\mathbf{x}) + \frac{1}{4}b_1\|\dot{\mathbf{x}}\|^2$$

holds for all $\mathbf{x}$ from the above-mentioned neighbourhood.

Differentiating the function V_2, by virtue of (2.6), we obtain that the inequality

$$\left.\frac{dV_2}{dt}\right|_{(2.6)} \leq -\theta(b_0 - \Delta_2)\|G_{\mathbf{x}}\|^2 - \left(\frac{a_0b_1^2}{2} - b_2\Delta_1\right)\|\dot{\mathbf{x}}\|^2 + \\ + (b_2\Delta_2 + \theta(a_1 + \Delta_1))\|G_{\mathbf{x}}\|\|\dot{\mathbf{x}}\| + \theta\|G_{\mathbf{xx}}\|\|\dot{\mathbf{x}}\|^2$$

holds when $t \geq 0$, $||\mathbf{x}|| < H$, $\mathbf{x} \in \mathbf{E}^n$.

If the parameters Δ_1 and Δ_2 satisfy the conditions

$$\Delta_2 < b_0, \quad 4(b_0 - \Delta_2)\left(\frac{a_0b_1^2}{2} - b_2\Delta_1\right) > \frac{(b_2\Delta_2 + \theta(a_1 + \Delta_1))^2}{\theta} \tag{2.9}$$

then the relation

$$\left.\frac{dV_2}{dt}\right|_{(2.6)} \leq -\alpha(\|G_{\mathbf{x}}\|^2 + \|\dot{\mathbf{x}}\|^2)$$

is satisfied for all $t \geq 0$, $\dot{\mathbf{x}} \in \mathbf{E}^n$ and sufficiently small values of $||\mathbf{x}||$. The positive constant α depends on the choice of the quantities Δ_1, Δ_2 and θ.

In order to complete the proof of the theorem, it remains to find a $\theta_0 > 0$ such that, when $\theta = \theta_0$, inequalities (2.9) give the largest domain of admissible values of Δ_1 and Δ_2. It is easily shown that $\theta_0 = b_2\Delta_2/(a_1 + \Delta_2)$ and conditions (2.9) when $\theta = \theta_0$ reduce to satisfying inequality (2.7).

3. Estimates of the solutions and stability conditions for perturbed systems

We will now consider the case when the function $G(\mathbf{x})$ in system (1.1) is a continuously differentiable, positive-definite, homogeneous function of order $\lambda + 1$, $\lambda \geq 1$. As before, we will assume that the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are bounded when $t \geq 0$. Furthermore, we will assume that the matrix $\mathbf{B}(t)$ is also bounded in the interval $[0, +\infty]$. The Lyapunov function for system (1.1) can then be chosen in the form

$$\tilde{V} = G(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}} + \theta\|\mathbf{x}\|^{\lambda-1}\mathbf{x}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}} \tag{3.1}$$

where θ is a positive constant. If the value of θ is sufficiently small, numbers Δ, β_1, β_2, $\beta_3 > 0$ will exist such that the relations

$$\beta_1(\|\mathbf{x}\|^{\lambda+1} + \|\dot{\mathbf{x}}\|^2) \le \tilde{V} \le \beta_2(\|\mathbf{x}\|^{\lambda+1} + \|\dot{\mathbf{x}}\|^2)$$
$$\left.\frac{d\tilde{V}}{dt}\right|_{(1.1)} \le -\beta_3(\|\mathbf{x}\|^{2\lambda} + \|\dot{\mathbf{x}}\|^2) \tag{3.2}$$

hold when $||\mathbf{x}|| < \Delta$ and for all $t \ge 0$, $\dot{\mathbf{x}} \in \mathbf{E}^n$.

Using inequalities (3.2), we obtain (Ref. 15, pp. 35–39, and Ref. 18) that the following theorems hold.

Theorem 2. *Suppose matrix (2.3) is positive-definite. Then, if* $\lambda = 1$, *it is possible to find numbers* $\gamma_1, \gamma_2 > 0$ *such that*

$$\|\mathbf{x}(t)\| + \|\dot{\mathbf{x}}(t)\| \le \gamma_1(\|\mathbf{x}(t_0)\| + \|\dot{\mathbf{x}}(t_0)\|)\exp(-\gamma_2(t - t_0))$$

for any solution $\mathbf{x}(t)$ *of system (1.1) and all* $t \ge t_0 \ge 0$, *that is, the equilibrium position (1.2) is exponentially stable and, if* $\lambda > 1$, *numbers* δ, γ_3, $\gamma_4 > 0$ *exist such that the estimates*

$$t_0 \ge 0, \quad \|\mathbf{x}(t_0)\| < \delta, \quad \|\dot{\mathbf{x}}(t_0)\| < \delta$$

for the solutions $\mathbf{x}(t)$ *of system (1.1) with initial data which satisfy the conditions*

$$\|\mathbf{x}(t)\|^{\lambda+1} + \|\dot{\mathbf{x}}(t)\|^2 \le \gamma_3\left(1 + \gamma_4(\|\mathbf{x}(t_0)\|^{\lambda+1} + \|\dot{\mathbf{x}}(t_0)\|^2)^{\frac{\lambda-1}{\lambda+1}}(t - t_0)\right)^{-\frac{\lambda+1}{\lambda-1}}$$

will hold for all $t \ge t_0$.

Theorem 3. *Suppose the perturbed system*

$$\ddot{\mathbf{x}} + \mathbf{A}(t)\dot{\mathbf{x}} + \mathbf{B}(t)G_{\mathbf{x}} = \mathbf{R}(t, \mathbf{x}, \dot{\mathbf{x}}) \tag{3.3}$$

is given, where the vector function $\mathbf{R}(t, \mathbf{x}, \dot{\mathbf{x}})$ *is continuous in the domain* $t \ge 0$, $||\mathbf{x}|| < H$, $||\dot{\mathbf{x}}|| < H$ *and satisfies the condition*

$$\|\mathbf{R}(t, \mathbf{x}, \dot{\mathbf{x}})\| \le L(\|\mathbf{x}\|^{\eta} + \|\dot{\mathbf{x}}\|^{\sigma}), \quad L, \eta, \sigma > 0$$

Then, if matrix (2.3) is positive-definite, the equilibrium position (1.2) of system (3.3) is asymptotically stable when the inequalities $\eta > \lambda$, $\sigma > 1$ *are satisfied.*

Remarks.

5°. In the proof of Theorems 2 and 3, instead of Lyapunov function (2.4), it is necessary to use a function constructed using formula (3.1) since the estimates of the rates, obtained using the function $\tilde{V}$, at which the solutions of system (1.1) tend to the equilibrium position and, also, the conditions, for which, when satisfied, the perturbations do not disturb the asymptotic stability, are more accurate. We also note that, when the Lyapunov function is constructed in the form of (3.1), it is not required that the function $G(\mathbf{x})$ should be doubly continuously differentiable. However, in this case, the homogeneity of the function $G(\mathbf{x})$ and the boundedness of the matrix $\dot{\mathbf{B}}(t)$ are used.

6°. The Lyapunov function $\tilde{V}$ which has been constructed and inequalities (3.2) established for it enable one, using the specified matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$, the function $G(\mathbf{x})$ and the fixed number θ, to obtain the actual values of the constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ occurring in the estimates of the solutions of system (1.1) which are mentioned in Theorem 2.

7°. If, as in the preceding section, it is assumed that the function $G(\mathbf{x})$ is doubly continuously differentiable, then the Lyapunov function for system (1.1) can be chosen in the form of (2.8). Using this function, we obtain that Theorems 2 and 3 will also hold without the additional condition of the boundedness of the matrix $\dot{\mathbf{B}}(t)$.

4. Conditions of asymptotic stability with respect to some of the variables

Now suppose the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are unbounded in the interval $[0, +\infty)$. We shall show that, in this case, the method for constructing Lyapunov functions proposed here enables one to obtain sufficient conditions for the asymptotic stability of the equilibrium position with respect of some of the variables.

Consider the matrix

$$\mathbf{D}(t) = \frac{1}{2}\mathbf{B}^{-1/2}(t)(\dot{\mathbf{B}}(t) + \mathbf{A}(t)\mathbf{B}(t) + \mathbf{B}(t)\mathbf{A}^T(t))\mathbf{B}^{-1/2}(t)$$

Theorem 4. *Suppose the matrix* $\mathbf{D}(t)$ *is positive-definite and numbers* $M > 0$ *and* $N > 0$, *exist such that the inequalities*

$$\|\mathbf{B}^{-1}(t)(\dot{\mathbf{B}}(t)\mathbf{B}^{-1/2}(t) + \mathbf{A}(t)\mathbf{B}^{1/2}(t))\mathbf{D}^{-1/2}(t)\| \le M$$
$$\|\mathbf{B}^{-1/2}(t)\mathbf{D}^{-1/2}(t)\|\,\|\mathbf{B}^{1/2}(t)\mathbf{D}^{-1/2}(t)\| \le N$$

are satisfied for all $t \ge 0$. *The equilibrium position* (1.2) *of system* (1.1) *is then asymptotically stable with respect to* $\mathbf{x}$.

Proof. We construct the Lyapunov function in the form

$$V = G(\mathbf{x}) + \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}} + G(\mathbf{x})\dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)G_{\mathbf{x}}$$

We obtain that a number $\delta > 0$ can be chosen such that the relations

$$\begin{aligned}&\alpha_1(G(\mathbf{x}) + \dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}}) \le V \le \alpha_2(G(\mathbf{x}) + \dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}})\\ &\left.\frac{dV}{dt}\right|_{(1.1)} \le -\alpha_3(G(\mathbf{x})\|G_{\mathbf{x}}\|^2 + \dot{\mathbf{x}}^T\mathbf{B}^{-1}(t)\dot{\mathbf{x}})\end{aligned} \tag{4.1}$$

hold when $||\mathbf{x}|| < \delta$ and for all $t \ge 0$, $\dot{\mathbf{x}} \in \mathbf{E}^n$. Here, α_1, α_2, α_3 are positive constants.

We specify that $\varepsilon > 0$ and $t_0 \ge 0$ and we shall assume that $\varepsilon < \delta$. Suppose

$$\mathbf{z} = (z_1, \dots, z_n)^T, \quad \lambda = \min_{\|\mathbf{x}\|^2 + \|\mathbf{z}\|^2 = \varepsilon^2} \alpha_1(G(\mathbf{x}) + \|\mathbf{z}\|^2)$$

It follows from the positive definiteness of the function $G(\mathbf{x})$ that $\lambda > 0$.

For a specified $t_0 \ge 0$, we select a number $\delta_1 > 0$ such that, when the condition

$$\|\mathbf{x}_0\|^2 + \|\dot{\mathbf{x}}_0\|^2 < \delta_1^2$$

is satisfied, the following inequalities hold

$$\|\mathbf{x}_0\|^2 + \dot{\mathbf{x}}_0^T\mathbf{B}^{-1}(t_0)\dot{\mathbf{x}}_0 < \varepsilon^2, \quad \alpha_2(G(\mathbf{x}_0) + \dot{\mathbf{x}}_0^T\mathbf{B}^{-1}(t_0)\dot{\mathbf{x}}_0) < \lambda$$

Using the estimates (4.1), we obtain that, if the relation

$$\|\mathbf{x}(t_0)\|^2 + \|\dot{\mathbf{x}}(t_0)\|^2 < \delta_1^2 \tag{4.2}$$

holds for the solution $\mathbf{x}(t)$ of system (1.1) when $t = t_0$, then

$$||\mathbf{x}(t)||^2 + \dot{\mathbf{x}}^T(t)\mathbf{B}^{-1}(t)\dot{\mathbf{x}}(t) < \varepsilon^2 \text{ for all } t \ge t_0$$

This means that the equilibrium position (1.2) is stable with respect to $\mathbf{x}$.

To prove asymptotic $\mathbf{x}$-stability, we will consider the solution $\mathbf{x}(t)$ with initial data which satisfy condition (4.2) and show that

$$\tilde{V}(t) = V(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) \to 0 \;\; \textbf{when} \;\; t \to +\infty$$

The function $\tilde{V}(t)$ is non-negative and decreases monotonically in the interval $[t_0, +\infty)$. Hence, $\lim_{t\to+\infty}\tilde{V}(t) = \eta \geq 0$ therefore exists. If $\eta > 0$, then, for all $t \geq t_0$, we have $\dot{\tilde{V}}(t) \leq -\beta$, where β is a positive constant. Integrating this inequality, we obtain that the relations

$$0 \leq \tilde{V}(t) \leq \tilde{V}(t_0) - \beta(t - t_0) \text{ when } t \geq t_0$$

hold and we arrive at a contradiction.

Corollary. *Suppose the scalar equation*

$$\ddot{x} + a(t)\dot{x} + b(t)g(x) = 0 \tag{4.3}$$

is given, where the function $a(t)$ *is continuous and* $b(t)$ *is continuously differentiable in the interval* $[0, +\infty)$, *and the function* $g(x)$ *is continuously differentiable when* $|x| < H$ *and satisfies the condition* $xg(x) > 0$ *when* $x \neq 0$. *We now introduce the notation* $\beta(t) = \dot{b}(t)/b(t)$. *If numbers* a_0, b_0, $M > 0$ *exist such that the inequalities*

$$b(t) \geq b_0, \quad \beta(t)/2 + a(t) \geq a_0, \quad (\beta(t) + a(t))^2 \leq Mb(t)(\beta(t)/2 + a(t))$$

are satisfied for all $t \geq 0$, *then the equilibrium position*

$$x = \dot{x} = 0 \tag{4.4}$$

of Eq. (4.3) *is asymptotically stable with respect to x.*

5. Application of differential inequalities

We will now investigate the scalar Eq. (4.3) in greater detail. Again, we will assume that the function $a(t)$ is continuous and that $b(t)$ is continuously differentiable in the interval $[0, +\infty)$. Suppose $b(t) > 0$ when $t \geq 0$. Regarding the function $g(x)$, we shall now assume that it is defined and continuous for all $x \in (-\infty, +\infty)$ and possesses the property

$$G(x) = \int_0^x g(\tau)d\tau > 0 \text{ when } x \neq 0$$

In particular, it follows from this conditions that $g(0) = 0$. Hence, the equation being considered, as previously, has the equilibrium position (4.4). In this case, this equilibrium position cannot be isolated.

An approach, based on the use of the theory of differential inequalities of the Chaplygin type, has been proposed[1] for obtaining the conditions for the stability of the equilibrium position of Eq. (4.3). The Lyapunov function for the equation being investigated was chosen in the form

$$V = b(t)G(x) + \dot{x}^2/2 \tag{5.1}$$

In the case of this function, the estimate

$$\left.\frac{dV}{dt}\right|_{(4.3)} \leq ([\beta(t)]_+ + 2[a(t)]_-)V \tag{5.2}$$

where

$$[\beta(t)]_+ = \max\{0; \beta(t)\}, \quad [a(t)]_- = \max\{0; -a(t)\}$$

holds for all $\dot{\mathbf{x}} \in (-\infty, +\infty)$ and $t \geq 0$. Using differential inequalities (5.2), it has been proved[1] that, when the conditions

$$\int_0^{+\infty} [\beta(t)]_+ dt < +\infty, \quad \int_0^{+\infty} [a(t)]_- dt < +\infty$$

are satisfied, the equilibrium position (4.4) of Eq. (4.3) is stable with respect to $\dot{x}$ and, if a number $b_0 > 0$ exists such that $b(t) \geq b_0$ when $t \geq 0$, then the equilibrium position is stable with respect to all the variables.

We shall show that, in the case of Lyapunov function (5.1), it is possible to obtain a more accurate estimate compared with estimate (5.2). Actually, we shall consider the function $c(t) = \max\{\beta(t); -2a(t)\}$. For any $x, \dot{x} \in (-\infty, +\infty)$ and $t \geq 0$, we have

$$\left.\frac{dV}{dt}\right|_{(4.3)} \leq c(t)V \tag{5.3}$$

Introducing the notation

$$\hat{c}(t_1, t_2) = \int_{t_1}^{t_2} c(\tau)d\tau$$

we therefore conclude that the following theorem holds.

Theorem 5. *Suppose* $\hat{c}(0, t) \leq M$ *for all* $t \in [0, +\infty)$, *where* M *is a certain constant. The equilibrium position* (4.4) *of* Eq. (4.3) *is then stable with respect to* $\dot{x}$ *and, if a number* $b_0 > 0$ *exists such that* $b(t) \geq b_0$ *when* $t \geq 0$, *then the equilibrium position is stable with respect to all the variables.*

Theorem 5 imposes less rigorous constraints on the functions $a(t)$ and $b(t)$ compared with the constraints obtained earlier in Ref. 1. For example, if $a(t) = \cos t$, $b(t) = e - 2r$, then the condition

$$\int_0^{+\infty} [a(t)]\_dt < +\infty$$

is not satisfied and Theorem 5 guarantees the stability of the equilibrium position with respect to $\dot{x}$.

Unlike estimate (5.2), estimate (5.3) enables us to obtain sufficient conditions for the asymptotic stability of the equilibrium position with respect to some of the variables.

Theorem 6. *Suppose*

$$\lim_{t \to +\infty} \hat{c}(0, t) = -\infty \tag{5.4}$$

The equilibrium position (4.4) *of* Eq. (4.3) *is then asymptotically stable with respect to* $\dot{x}$.

Using estimate (5.3), it is possible to determine the conditions for the stability of the equilibrium position with respect to all the variables and the asymptotic stability with respect to $\dot{x}$.

In fact, integrating inequality (5.3), we obtain that the relations

$$G(x(t)) \leq \frac{1}{b(t)}\exp(\hat{c}(t_0, t))\left(b(t_0)G(x_0) + \frac{\dot{x}_0^2}{2}\right)$$

$$\dot{x}^2(t) \leq \exp(\hat{c}(t_0, t))(2b(t_0)G(x_0) + \dot{x}_0^2)$$

hold for all $t \geq t_0$ in the case of a solution $x(t)$ of Eq. (4.3) with initial data

$$t_0 \geq 0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0$$

On the other hand, when $t \geq t_0$, we have

$$|x(t)| \leq |x_0| + \int_{t_0}^{t} |\dot{x}(\tau)|d\tau \leq$$

$$\leq |x_0| + (2b(t_0)G(x_0) + \dot{x}_0^2)^{1/2}\int_{t_0}^{t} \exp\left(\frac{1}{2}\hat{c}(t_0, \tau)\right)d\tau$$

We therefore obtain the following theorem.

Theorem 7. *Suppose the limiting relation holds. Then, if just one of the conditions*

a) $\int_0^{+\infty} \exp\left(\frac{1}{2}\hat{c}(0,\tau)d\tau\right) d\tau < +\infty$

b) $\exp(\hat{c}(0,t)) \le Lb(t)$ *when* $t \ge 0$, *where L is a positive constant*

is satisfied, the equilibrium position (4.4) *of* Eq. (4.3) *is stable with respect to all the variables and asymptotically stable with respect to* $\dot{x}$.

In particular, condition *b* is satisfied if $\beta(t) \ge -2a(t)$ for all $t \ge 0$ and, if numbers $a_0 > 0$ and $\bar{b} > 0$ exists such that the inequalities

$$a(t) \ge a_0, \quad \beta(t) \le -\bar{b} \tag{5.5}$$

hold in the interval $[0, +\infty)$, then condition *a* is satisfied.

Remarks.

8°. Conditions *a* and *b* of Theorem 7 can be satisfied independently of one another. For example, if $a(t) = 1$, $b(t) = e^{-3t}$, then condition *a* is satisfied and condition *b* is not. On the other hand, if $a(t) = 1$, $b(t) = 1/(t+1)$, condition *b* is satisfied and condition *a* is not.

9°. Using the method of μ-systems, sufficient conditions for stability with respect to all the variables and exponential stability with respect to $\dot{x}$ have been obtained in the case of the linear oscillator (2.1) (Ref. 19, pp. 264, 265). However, these conditions are cruder compared with the conditions which can be obtained using Theorem 7. In addition to inequalities (5.5) being satisfied, it was required in the above mentioned paper that the relation

$$-a(t)\beta(t) - (1 - b(t))^2/4 \ge \varepsilon = \text{const} > 0 \tag{5.6}$$

should hold for all $t \ge 0$.

For example, if $a(t) = a_0$, $b(t) = b_0 e^{-\delta t}$, where a_0, b_0, δ are positive constants, then inequalities (5.5) will be satisfied for any values of a_0, b_0 and δ, and condition (5.6) leads to the additional constraint: $a_0\delta > 1/4$.

The theorems which have been proved in this section can also be extended to systems of the form

$$\ddot{\mathbf{x}} + \mathbf{A}(t)\dot{\mathbf{x}} + b(t)G_{\mathbf{x}} = \mathbf{0} \tag{5.7}$$

where $\mathbf{x}$ is an n-dimensional vector, the elements of the matrix $\mathbf{A}(t)$ are given and continuous when $t \ge 0$, the scalar function $b(t)$ is continuously differentiable and positive in the interval $[0, +\infty)$ and the function $G(\mathbf{x})$ is defined and continuously differentiable for all $\mathbf{x} \in \mathbf{E}^n$. Moreover, $G(\mathbf{x}) > 0$ when $\mathbf{x} \ne \mathbf{0}$, $G(\mathbf{0}) = 0$.

To do this, it is necessary to select the function

$$V = b(t)G(\mathbf{x}) + \dot{\mathbf{x}}^T\dot{\mathbf{x}}/2$$

as the Lyapunov function. We obtain

$$\left.\frac{dV}{dt}\right|_{(5.7)} = \dot{b}(t)G(\mathbf{x}) - \dot{\mathbf{x}}^T\mathbf{A}(t)\dot{\mathbf{x}}$$

Suppose

$$a(t) = \min_{j = 1, \dots, n} \lambda_j(t)$$

Here, $\lambda_j(t)$ are the eigenvalues of the matrix $(\mathbf{A}(t) + \mathbf{A}^T(t))/2$. Using the function $a(t)$, we again arrive at a differential inequality of the form of (5.3).

Cases were also considered when investigating the stability of the equilibrium positions of equations of the form of (4.3) when the generalized stiffness can take negative values (Refs. 1,5, pp. 175–178 and 19, pp. 264, 265). We shall show that, if $b(t) < 0$ for all $t \geq 0$, then the approach proposed in this paper also enables one to obtain new conditions for the stability of an equilibrium position.

As the Lyapunov functions for Eq. (4.3), we select the function

$$\tilde{V} = -b(t)G(x) + \dot{x}^2/2$$

We have

$$\left.\frac{d\tilde{V}}{dt}\right|_{(4.3)} = -\dot{b}(t)G(x) - a(t)\dot{x}^2 - 2b(t)g(x)\dot{x}$$

We shall assume that the inequality $g^2(x) \leq KG(x)$ holds in a certain neighbourhood of the point $x = 0$, where K is a positive constant. For example, this inequality holds in the case when $g(x) = x^{\mu}$, where μ is a rational number with an odd numerator and odd denominator, $\mu \geq 1$.

The relation

$$\left.\frac{d\tilde{V}}{dt}\right|_{(4.3)} = -\beta(t)b(t)G(x) - a(t)\dot{x}^2 + \sqrt{-b(t)}(-Kb(t)G(x) + \dot{x}^2)$$

is then satisfied for all $t \geq 0$, $\dot{x} \in (-\infty, +\infty)$ and values of x which are sufficiently small in modulus, and, using this relation, we obtain the following theorem

Theorem 8. *If $b(t) < 0$ when $t \geq 0$ and numbers $a_0 > 0$ and $\bar{b} > 0$ exist such that inequalities (5.5) hold in the interval* $[0, +\infty)$, *then the equilibrium position (4.4) of Eq. (4.3) is stable with respect to all the variables and exponentially stable with respect to $\dot{x}$.*

Remark.

10°. The constraints on the functions $a(t)$ and $b(t)$ formulated in Theorem 8 are less rigid compared with the constraints obtained for the linear oscillator (2.1) (Ref. 19, pp. 264, 265).

6. Extension to cases of non-linear dissipative-accelerating forces

We will now show that the approach considered in the preceding section can be extended to systems with non-linear dissipative-accelerating forces.

Suppose we are given a Liénard equation with variable parameters[20–23]

$$\ddot{x} + \frac{d}{dt}(a(t)f(x)) + b(t)g(x) = 0 \tag{6.1}$$

Here, the functions $a(t)$ and $b(t)$ are continuously differentiable in the interval $[0, +\infty)$ and $b(t) > 0$ for all $t \geq 0$. The function $f(x)$ is continuously differentiable and the function $g(x)$ is continuous in a certain neighbourhood of the point $x = 0$.

We will assume that, in the above mentioned neighbourhood, the functions $f(x)$ and $g(x)$ can be represented in the form

$$f(x) = x^{\nu+1} + \tilde{f}(x), \qquad g(x) = x^{\lambda} + \tilde{g}(x)$$

where ν is a positive rational number with an even numerator and an odd denominator, λ is a positive rational number with an odd numerator and an odd denominator and

$$\tilde{f}(x)/x^{\nu+1} \to 0, \quad \tilde{g}(x)/x^{\lambda} \to 0 \textbf{ when } x \to 0$$

Then, the equation

$$\ddot{x} + \frac{d}{dt}(a(t)x^{\nu+1}) + b(t)x^{\lambda} = 0$$

can be considered as the equation for the non-linear approximation of (6.1). This equation is equivalent to the system

$$\dot{x} = y - a(t)x^{\nu+1}, \quad \dot{y} = -b(t)x^{\lambda} \tag{6.2}$$

In order to investigate the stability of the equilibrium position

$$x = y = 0 \tag{6.3}$$

of this system, we choose the Lyapunov function in the form

$$V = \frac{x^{\lambda+1}}{\lambda+1} + \frac{y^2}{2b(t)}$$

We obtain

$$\left.\frac{dV}{dt}\right|_{(6.2)} = -a(t)x^{\lambda+\nu+1} - \frac{\beta(t)y^2}{2b(t)}$$

We now choose a number $\Delta > 0$. Then, the inequality

$$\left.\frac{dV}{dt}\right|_{(6.2)} \le \hat{c}(t)V$$

holds when $|x| \le \Delta$ and for all $y \in (-\infty, +\infty)$, $t \ge 0$, where

$$\hat{c}(t) = \max\{[\beta(t)]_{-}; (\lambda+1)\Delta^{\nu}[a(t)]_{-}\}$$

This means that, if the solution $(x(t), y(t))^T$ of system (6.2) in a certain interval $[t_0, t_1]$ satisfies the condition $|x(t)| \le \Delta$, then the relation

$$V(t, x(t), y(t)) \le V(t_0, x(t_0), y(t_0))\exp\left(\int_{t_0}^{t}\hat{c}(\tau)d\tau\right) \tag{6.4}$$

is satisfied in the above mentioned interval. We obtain the following theorem.

Theorem 9. *Suppose*

$$\int_0^{+\infty}[\beta(t)]_{-}dt < +\infty, \quad \int_0^{+\infty}[a(t)]_{-}dt < +\infty$$

Then, the equilibrium position (6.3) of system (6.2) is stable with respect to x and, if the function b(t) is bounded in the interval $[0, +\infty)$, *then the equilibrium position is stable with respect to all of the variables.*

We will now assume that $\dot{b}(t) \ge 0$ when $t \ge 0$. In this case, Theorem 9 can be strengthened.

Again, we choose a number $\Delta > 0$. If the inequality $y^2/(2b(t)) \le \Delta$ is satisfied, then the following relations hold

$$\left.\frac{dV}{dt}\right|_{(6.2)} \le \varphi_1(t)\left(\frac{x^{\lambda+1}}{\lambda+1}\right)^{1+\frac{\nu}{\lambda+1}} + \varphi_2(t)\left(\frac{y^2}{2b(t)}\right)^{1+\frac{\nu}{\lambda+1}} \le \tilde{c}(t)V^{1+\frac{\nu}{\lambda+1}}$$

Here,

$$\varphi_1(t) = -(\lambda+1)^{1+\nu/(\lambda+1)}a(t), \quad \varphi_2(t) = -\Delta^{-\nu/(\lambda+1)}\beta(t)$$

$$\tilde{c}(t) = \max_{u_1 \ge 0,\, u_2 \ge 0,\, u_1+u_2=1}\left(\varphi_1(t)u_1^{1+\frac{\nu}{\lambda+1}} + \varphi_2(t)u_2^{1+\frac{\nu}{\lambda+1}}\right)$$

It is easy to show that $\tilde{c}(t) = \varphi_1(t)$ if $\varphi_1(t) \geq 0$ and

$$\tilde{c}(t) = -\varphi_1(t)\varphi_2(t)(|\varphi_1(t)|^{(\lambda+1)/\nu} + |\varphi_2(t)|^{(\lambda+1)/\nu})^{-\nu/(\lambda+1)}$$

if $\varphi_1(t) \geq 0$. We obtain that the estimate

$$V(t, x(t), y(t)) \leq \left(V^{-\frac{\nu}{\lambda+1}}(t_0, x(t_0), y(t_0)) - \frac{\nu}{\lambda+1}\int_{t_0}^{t}\tilde{c}(\tau)d\tau\right)^{-\frac{\lambda+1}{\nu}} \tag{6.5}$$

holds for the solution $(x(t), y(t))^T$ of system (6.2) which satisfies the condition $y^2(t)/(2b(t)) \leq \Delta$ in a certain interval t_0, t_1 for all $t \in [t_0, t_1]$.

In this case, the initial data $x(t_0)$, $y(t_0)$ of the solution being considered must be sufficiently small in order that the relation

$$\nu V^{\nu/(\lambda+1)}(t_0, x(t_0), y(t_0))\int_{t_0}^{t}\tilde{c}(\tau)d\tau < \lambda + 1$$

is satisfied in the above mentioned interval.

The following theorems therefore hold.

Theorem 10. *Suppose the inequalities $\dot{b}(t) \geq 0$ and*

$$\int_0^t \tilde{c}(\tau)d\tau \leq M \tag{6.6}$$

hold for all $t \geq 0$, where m is a certain constant. Then, the equilibrium position (6.3) of system (6.2) is stable with respect to x and, if the function $b(t)$ is bounded in the interval $[0, +\infty)$, then the equilibrium position is stable with respect to all the variables.

Theorem 11. *Suppose $\dot{b}(t) \geq 0$ when $t \geq 0$ and the following limiting relation holds*

$$\lim_{t\to+\infty}\int_0^t \tilde{c}(\tau)d\tau = -\infty \tag{6.7}$$

Then, the equilibrium position (6.3) of system (6.2) is asymptotically stable with respect to x.

Remarks.

11°. Estimates (6.4) and (6.5) will be more accurate, the smaller the value of Δ which is chosen. However, by reducing the parameter Δ, we narrow the domain of the initial data of the solutions for which it is possible to make use of these estimates.

12°. The function $\tilde{c}(t)$ depends on the choice of the parameter Δ. Here, conditions (6.6) and (6.7) can be satisfied for some values of Δ and not for others. Theorems 10 and 11 can be used if these conditions are satisfied just for sufficiently small values of Δ.

13°. If the function $\dot{b}(t)$ when $t \in [0, +\infty)$ takes values with different signs, it is necessary to use inequality (6.4) to estimate the solutions of system (6.2) in those intervals in which $\dot{b}(t) < 0$ and inequality (6.5) in the intervals in which $\dot{b}(t) \geq 0$.

The results obtained for the non-linear approximation equation can also be extended to the initial equation (6.1). Actually, we change from Eq. (6.1) to the system

$$\dot{x} = y - a(t)f(x), \quad \dot{y} = -b(t)g(x) \tag{6.8}$$

which is equivalent to it, for which we choose the Lyapunov function in the form

$$V = \int_0^x g(\tau)d\tau + \frac{y^2}{2b(t)}$$

Then,

$$\left.\frac{dV}{dt}\right|_{(6.8)} = -a(t)g(x)f(x) - \frac{\beta(t)y^2}{2b(t)}$$

In this case, a $\delta > 0$ exists for any number $\varepsilon \in (0, 1)$ such that when $|x| \le \delta$ the following limits hold

$$\frac{(1-\varepsilon)x^{\lambda+1}}{\lambda+1} \le \int_0^x g(\tau)d\tau \le \frac{(1+\varepsilon)x^{\lambda+1}}{\lambda+1}$$
$$(1-\varepsilon)x^{\lambda+\nu+1} \le g(x)f(x) \le (1+\varepsilon)x^{\lambda+\nu+1} \tag{6.9}$$

Using the limits (6.9), theorems analogous to Theorems 9, 10 and 11 can be proved in the case of system (6.8).

7. Uniaxial stabilization of a rigid body

We will now consider the use of the theorems which have been proved above to solve the problem of the uniaxial stabilization of a rigid body.[15,24]

Suppose a rigid body is specified which is rotating about a fixed point O. The dynamic Euler's equations, describing the rotational motion of the body under the action of a moment **M**, have the form

$$(\mathbf{I}^{\boldsymbol{\cdot}}\omega) + \omega \times \mathbf{I}\omega = \mathbf{M} \tag{7.1}$$

Here ω is the angular velocity vector and **I** is the inertia tensor of the body.[25,26]

Consider the two unit vectors **r** and **s** and assume that the vector **s** is invariant in absolute space and that the vector **r** is invariant in the rigid body. Consequently, the vector **s** rotates with respect to the rectangular system of coordinates $Oxyz$ associated with the body with an angular velocity $-\omega$. We obtain

$$\dot{\mathbf{s}} = -\omega \times \mathbf{s} \tag{7.2}$$

Suppose the moment **M** is defined by the formula

$$\mathbf{M} = -\mathbf{A}(t)\omega - b(t)\mathbf{s} \times \mathbf{r} \tag{7.3}$$

where $\mathbf{A}(t)$ is a continuous matrix when $t \ge 0$ and $b(t)$ is a positive and continuously differentiable scalar function when $t \ge 0$. An equilibrium position

$$\omega = \mathbf{0}, \quad \mathbf{s} = \mathbf{r} \tag{7.4}$$

then exists in the case of the system of Eqs. (7.1), (7.2).

It is required to determine what conditions the matrix $\mathbf{A}(t)$ and the function $b(t)$ must satisfy in order that this equilibrium position should be asymptotically stable with respect to all or some of the variables.

We will first investigate the case when the body has variable moments of inertia.[26,27]. We shall assume that $\mathbf{I}(t)$ is a continuously differentiable, bounded and positive-definite matrix and that $b(t) \equiv b = \text{const} > 0$.

In accordance with the approach proposed in Section 2, we choose the Lyapunov function in the form

$$V_1 = \frac{b}{2}\|\mathbf{s}-\mathbf{r}\|^2 + \frac{1}{2}\omega^T\mathbf{I}(t)\omega + \theta\omega^T\mathbf{I}(t)\mathbf{s} \times \mathbf{r}, \quad \theta > 0$$

It is easy to show that, if the matrix

$$\mathbf{C}(t) = \mathbf{A}(t) + \mathbf{A}^T(t) + \dot{\mathbf{I}}(t)$$

is positive-definite and the matrix $\mathbf{A}(t)$ is bounded when $t \geq 0$, then, for sufficiently small values of θ, the function V_1 satisfies the requirements of the Lyapunov theorem on asymptotic stability. Consequently, the equilibrium position (7.4) will be asymptotically stable with respect to all variables.

If together with the moment **M**, a moment of the perturbing forces **M** acts on the body for which the inequality

$$\|\mathbf{M}_1\| \leq \Delta_1\|\omega\| + \Delta_2\|\mathbf{s}-\mathbf{r}\|$$

holds in a certain neighbourhood of the equilibrium position, where Δ_1 and Δ_2 are positive constants, then, using the Lyapunov function which has been constructed as in the proof of Theorem 1, it is possible to obtain estimates of permissible perturbations which do not destroy the asymptotic stability of the equilibrium position.

Remarks.

14°. A similar method for constructing Lyapunov functions with negative-definite derivatives was used in Ref. 24 to solve problems of the uniaxial and triaxial stabilization of a rigid body. However, it was assumed in that case that the inertia tensor of the body and the control moment were independent of time.

15°. The conditions obtained above, which are imposed on the matrices $\mathbf{A}(t)$ and $\mathbf{I}(t)$, are identical to the conditions established in Ref. 27 when solving the problem of the triaxial stabilization of a rigid body with variable moments of inertia. However, in the above mentioned paper, a Lyapunov function with a derivative of fixed sign was used to prove the asymptotic stability.

We shall next assume that the inertial tensor of the body is a constant positive-definite matrix. As before, we assume that the moment **M** has the form of (7.3).

We now consider the function

$$V_2 = \frac{1}{2}\|\mathbf{s}-\mathbf{r}\|^2 + \frac{1}{2b(t)}\omega^T\mathbf{I}\omega + \frac{\theta}{b(t)}\omega^T\mathbf{I}\mathbf{s}\times\mathbf{r}, \quad \theta > 0$$

Suppose

$$\psi(t) = \min_{j=1,2,3} \lambda_j(t)$$

where $\lambda_j(t)$ are the eigenvalues of the matrix

$$\mathbf{A}(t) + \mathbf{A}^T(t) + \beta(t)\mathbf{I}$$

As in the proof of Theorem 4, it is possible to show using the Lyapunov function which has been constructed that, if positive constants a_0, b_0, R exist for which the inequalities

$$\psi(t) \geq a_0, \quad b(t) \geq b_0, \quad \|\mathbf{A}(t) + \beta(t)\mathbf{I}\| \leq Rb(t)\psi(t)$$

hold for all $t \geq 0$, then the equilibrium position (7.4) of system (7.1), (7.2) is asymptotically stable with respect to **s**.

We now select the Lyapunov function in the form

$$V_3 = b(t)\|\mathbf{s}-\mathbf{r}\|^2/2 + \omega^T\mathbf{I}\omega/2$$

Using the approach proposed in Section 5, we consider the function

$$c(t) = \max\{\beta(t); -a(t)\}$$

where $a(t)$ is the smallest eigenvalue of the matrix

$$\mathbf{I}^{-1/2}(\mathbf{A}(t) + \mathbf{A}^T(t))\mathbf{I}^{-1/2}$$

We obtain that, if the function $c(t)$ possesses the properties mentioned in Theorem 7, then the equilibrium position (7.4) of system (7.1), (7.2) is stable with respect to all the variables and asymptotically stable with respect to ω.

Hence, in the case when, in formula (7.3) which determines the moment **M** acting on the body, the function $b(t)$ satisfies the inequality $b_0 \leq b(t) \leq b_1$ when $t \geq 0$, where b_0 and b_1 are positive constants, the method considered in

Sections 2 and 3 can be used to construct the Lyapunov function. If the function $b(t)$ is non-zero $b(t) \geq b_0 = \text{const} > 0$ when $t \geq 0$ but, at the same time, is unbounded in the interval $[0, +\infty)$, then the approach proposed in Section 4 can be used to investigate the stability of the equilibrium position. But the method of differential inequalities can be used in the case when $b(t) \to 0$ when $t \to +\infty$, where we use the theorems proved in Section 5 in the case of a linear moment **M** and the approach considered in Section 6 in the case of a non-linear moment.

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